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Abstract

We characterize those non-negative, measurable functions ψ on $[0,1]$ and positive, continuous functions ω_1 and ω_2 on $\mathbb{R}^+$ for which the generalized Hardy-Cesàro operator

$$(U_\psi f)(x) = \int_0^1 f(tx)\psi(t)\,dt$$

defines a bounded operator $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$. This generalizes a result of Xiao ([7]) to weighted spaces. Furthermore, we extend U_ψ to a bounded operator on $M(\omega_1)$ with range in $L^1(\omega_2) \oplus \mathbb{C}\delta_0$, where $M(\omega_1)$ is the weighted space of locally finite, complex Borel measures on $\mathbb{R}^+$. Finally, we show that the zero operator is the only weakly compact generalized Hardy-Cesàro operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

1 Introduction

A classical result of Hardy ([5]) shows that the Hardy-Cesàro operator

$$(Uf)(x) = \frac{1}{x}\int_0^x f(s)\,ds$$

defines a bounded linear operator on $L^p(\mathbb{R}^+)$ with $\|U\| = p/(p-1)$ for $p > 1$. Clearly, U is not bounded on $L^1(\mathbb{R}^+)$. Hardy's result has been generalized in various ways, of which we will mention some, which have inspired this paper.

For $1 \le p \le q \le \infty$ and non-negative measurable functions u and v on $\mathbb{R}^+$, Muckenhoupt ([6]) and Bradley ([3]) gave a necessary and sufficient condition for the existence of a constant C such that

$$\left(\int_0^\infty \left(u(x)\int_0^x f(t)\,dt\right)^q dx\right)^{1/q} \le C\left(\int_0^\infty (v(x)f(x))^p\,dx\right)^{1/p}$$

for every positive, measurable function f on $\mathbb{R}^+$. This can be rephrased as a characterization of the weighted L^p and L^q spaces on $\mathbb{R}^+$ between which the Hardy-Cesàro operator U is bounded.

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In a different direction, for a non-negative measurable funtion ψ on $[0,1]$, Xiao ([7]) considered the generalized Hardy-Cesàro operators

$$(U_\psi f)(x) = \int_0^1 f(tx)\psi(t)\,dt$$

for measurable functions f on $\mathbb{R}^n$. We remark that

$$(U_\psi f)(x) = \frac{1}{x}\int_0^x f(s)\psi(s/x)\,ds$$

for measurable functions f on $\mathbb{R}$. Xiao proved that U_ψ defines a bounded operator on $L^p(\mathbb{R}^n)$ (for $p \geq 1$) if and only if

$$\int_0^1 \frac{\psi(t)}{t^{n/p}}\,dt < \infty.$$

Xiao's result is the main motivation for this paper.

Finally, we mention that Albanese, Bonet and Ricker in a recent series of papers (see, for instance, [1] and [2]) have considered the spectrum, compactness and other properties of the Hardy-Cesàro operator on various spaces of continuous functions and discrete spaces.

In this paper we will study the generalized Hardy-Cesàro operators between weighted spaces of integrable functions, and we will obtain a generalization of Xiao's result in this context. Let ω be a positive, continuous function on $\mathbb{R}^+$ and let $L^1(\omega)$ be the Banach space of (equivalence classes of) measurable functions f on $\mathbb{R}^+$ for which

$$\|f\|_{L^1(\omega)} = \int_0^\infty |f(t)|\omega(t)\,dt < \infty.$$

In the usual way we identify the dual space of $L^1(\omega)$ with the space $L^\infty(1/\omega)$ of measurable functions h on $\mathbb{R}^+$ for which

$$\|h\|_{L^\infty(1/\omega)} = \operatorname{ess\,sup}_{t\in\mathbb{R}^+}|h(t)|/\omega(t) < \infty.$$

We denote by $C_0(1/\omega)$ the closed subspace of $L^\infty(1/\omega)$ consisting of the continuous functions g in $L^\infty(1/\omega)$ for which g/ω vanishes at infinity. Finally, we identify the dual space of $C_0(1/\omega)$ with the space $M(\omega)$ of locally finite, complex Borel measures μ on $\mathbb{R}^+$ for which

$$\|\mu\|_{M(\omega)} = \int_{\mathbb{R}^+} \omega(t)\,d|\mu|(t) < \infty.$$

We consider the space $L^1(\omega)$ as a closed subspace of $M(\omega)$.

In Section 2 we characterize those functions ψ, ω_1 and ω_2 for which U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. These operators are extended to bounded operators on $M(\omega_1)$ in Section 3, where we also obtain results about their ranges. Finally, in Section 4 we show that there are no non-zero weakly compact generalized Hardy-Cesàro operators from $L^1(\omega_1)$ to $L^1(\omega_2)$.

2 A characterization of the generalized Hardy-Cesàro operators

For a non-negative, measurable function ψ on $[0,1]$ and positive, continuous functions ω_1 and ω_2 on $\mathbb{R}^+$, we say that condition (C) is satisfied if there exists a constant C such that

$$\int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt \le C\omega_1(s)$$

for every $s \in \mathbb{R}^+$.

Theorem 2.1 *Let ψ be a non-negative, measurable function on $[0,1]$ and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. Then U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$ if and only if condition (C) is satisfied.*

Proof Assume that condition (C) is satisfied and let $f \in L^1(\omega_1)$. Then

$$\int_0^\infty \int_0^1 |f(s)|\,\frac{\psi(t)}{t}\,\omega_2(s/t)\,dt\,ds \le C\int_0^\infty |f(s)|\omega_1(s)\,ds = C\|f\|_{L^1(\omega_1)} < \infty,$$

so it follows from Fubini's theorem that

$$\int_0^1 \int_0^\infty |f(tx)|\psi(t)\omega_2(x)\,dx\,dt = \int_0^1 \int_0^\infty |f(s)|\,\frac{\psi(t)}{t}\,\omega_2(s/t)\,ds\,dt \le C\|f\|_{L^1(\omega_1)} < \infty.$$

Another application of Fubini's theorem thus shows that $(U_\psi f)(x)$ is defined for almost all $x \in \mathbb{R}^+$ with

$$\begin{aligned}\|U_\psi f\|_{L^1(\omega_2)} &= \int_0^\infty |(U_\psi f)(x)|\omega_2(x)\,dx \le \int_0^\infty \int_0^1 |f(tx)|\psi(t)\omega_2(x)\,dt\,dx\\ &= \int_0^1 \int_0^\infty |f(tx)|\psi(t)\omega_2(x)\,dx\,dt \le C\|f\|_{L^1(\omega_1)} < \infty.\end{aligned}$$

Hence U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

Conversely, assume that U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Since $L^1(\omega_2)$ is a closed subspace of $M(\omega_2)$ which we identify with the dual space of $C_0(1/\omega_2)$, it follows from [4, Theorem VI.8.6] that there exists a map ρ from $\mathbb{R}^+$ to $M(\omega_2)$ for which the map $s \mapsto \langle g, \rho(s)\rangle = \int_{\mathbb{R}^+} g(x)\,d\rho(s)(x)$ is measurable and essentially bounded on $\mathbb{R}^+$ for every $g \in C_0(1/\omega_2)$ with $\|U_\psi\| = \operatorname{ess\,sup}_{s\in\mathbb{R}^+}\|\rho(s)\|_{M(\omega_2)}$ and such that

$$\langle g, U_\psi f\rangle = \int_0^\infty \langle g, \rho(s)\rangle f(s)\omega_1(s)\,ds = \int_0^\infty \int_{\mathbb{R}^+} g(x)\,d\rho(s)(x)\,f(s)\omega_1(s)\,ds$$

for every $g \in C_0(1/\omega_2)$ and $f \in L^1(\omega_1)$. On the other hand

$$\begin{aligned}\langle g, U_\psi f\rangle &= \int_0^\infty g(x)(U_\psi f)(x)\,dx\\ &= \int_0^\infty \int_0^x \frac{g(x)}{x}\,f(s)\psi(s/x)\,ds\,dx\\ &= \int_0^\infty \frac{1}{\omega_1(s)}\int_s^\infty \frac{g(x)}{x}\,\psi(s/x)\,dx\,f(s)\omega_1(s)\,ds\end{aligned}$$

for every $g \in C_0(1/\omega_2)$ and $f \in L^1(\omega_1)$, so it follows that

$$\int_{\mathbb{R}^+} g(x)\, d\rho(s)(x) = \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x}\, \psi(s/x)\, dx$$

for almost all $s \in \mathbb{R}^+$ and every $g \in C_0(1/\omega_2)$ (considering both sides as elements of $L^\infty(\mathbb{R}^+)$). Considered as elements of $M(\omega_2)$ we thus have

$$d\rho(s)(x) = \frac{1}{\omega_1(s)}\, \frac{1}{x}\, \psi(s/x)\, 1_{x \geq s}\, dx$$

for almost all $s, x \in \mathbb{R}^+$. Hence $\rho(s) \in L^1(\omega_2)$ with

$$\begin{aligned}
\|\rho(s)\|_{L^1(\omega_2)} &= \int_0^\infty \omega_2(x)\, d\rho(s)(x) \\
&= \frac{1}{\omega_1(s)} \int_0^\infty \frac{1}{x}\, \psi(s/x)\, 1_{x \geq s}\, \omega_2(x)\, dx \\
&= \frac{1}{\omega_1(s)} \int_s^\infty \frac{1}{x}\, \psi(s/x)\omega_2(x)\, dx \\
&= \frac{1}{\omega_1(s)} \int_0^1 \frac{\psi(t)}{t}\, \omega_2(s/t)\, dt
\end{aligned}$$

for almost all $s \in \mathbb{R}^+$. Therefore

$$\int_0^1 \omega_2(s/t)\, \frac{\psi(t)}{t}\, dt = \|\rho(s)\|_{L^1(\omega_2)}\omega_1(s) \leq \|U_\psi\|\omega_1(s)$$

for almost all $s \in \mathbb{R}^+$. Since both sides of the inequality are continuous functions of s, the inequality holds for every $s \in \mathbb{R}^+$, so condition (C) holds. $\square$

Letting $s = 0$ in condition (C) we see that Xiao's condition is necessary in our situation.

Corollary 2.2 *Let ψ be a non-negative, measurable function on $[0, 1]$ and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. If U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$, then*

$$\int_0^1 \frac{\psi(t)}{t}\, dt < \infty.$$

The following straightforward consequences can be deduced from Theorem 2.1.

Corollary 2.3 *Let ψ be a non-negative, measurable function on $[0, 1]$*

(a) Let ω be a decreasing, positive, continuous function on $\mathbb{R}^+$, and assume that $\int_0^1 \psi(t)/t\, dt < \infty$. Then U_ψ defines a bounded operator from $L^1(\omega)$ to $L^1(\omega)$.

(b) Let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$, and assume that ω_2 is increasing. If U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$, then there exists a constant C such that $\omega_2(s) \leq C\omega_1(s)$ for every $s \in \mathbb{R}^+$.

(c) Let ω be an increasing, positive, continuous function on $\mathbb{R}^+$, and assume that there exists $a < 1$ and $K > 0$ such that $\psi(t) \geq K$ almost everywhere on $[a, 1]$. If U_ψ defines a bounded operator from $L^1(\omega)$ to $L^1(\omega)$, then there exist positive constants C_1 and C_2 such that

$$C_1\omega(s) \leq \int_0^1 \omega(s/t)\,\frac{\psi(t)}{t}\,dt \leq C_2\omega(s)$$

for every $s \in \mathbb{R}^+$.

Proof (a): We have

$$\int_0^1 \omega(s/t)\,\frac{\psi(t)}{t}\,dt \leq \int_0^1 \frac{\psi(t)}{t}\,dt\,\omega(s)$$

for every $s \in \mathbb{R}^+$, so condition (C) is satisfied with $\omega_1 = \omega_2 = \omega$ and the result follows.

(b): We have

$$\int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt \geq \int_0^1 \frac{\psi(t)}{t}\,dt\,\omega_2(s)$$

for every $s \in \mathbb{R}^+$. Since condition (C) is satisfied, the result follows.

(c): We have

$$\int_0^1 \omega(s/t)\,\frac{\psi(t)}{t}\,dt \geq K\int_a^1 \omega(s/t)\,dt \geq K(1-a)\omega(s)$$

for every $s \in \mathbb{R}^+$. The other inequality is just condition (C) with $\omega_1 = \omega_2 = \omega$. □

We finish the section with some examples of functions ψ, ω_1 and ω_2 for which U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

Example 2.4

(a) For $\alpha > 0$, let $\psi(t) = t^\alpha$ for $t \in [0, 1]$. Also, for $\beta_1, \beta_2 \in \mathbb{R}$, let $\omega_i(x) = (1 + x)^{\beta_i}$ for $x \in \mathbb{R}^+$ and $i = 1, 2$. Then U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$ if and only if $\beta_2 \leq \beta_1$ and $\beta_2 < \alpha$.

(b) For $\alpha > 0$, let $\psi(t) = t^\alpha$ for $t \in [0, 1]$. Also, let $\omega_1(x) = e^{-x}/(1 + x)$ and $\omega_2(x) = e^{-x}$ for $x \in \mathbb{R}^+$. Then U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Moreover, it is not possible to replace $\omega_1(x)$ by a function tending faster to zero at infinity.

(c) Let $\psi(t) = e^{-1/t^2}$ for $t \in [0, 1]$. Also, let $\omega_1(x) = e^{x^2/4}/x$ and $\omega_2(x) = e^x$ for $x \in \mathbb{R}^+$. Then U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Moreover, it is not possible to replace $\omega_1(x)$ by a function tending slower to infinity at infinity.

Proof (a): For $s \geq 1$ and $t \in [0, 1]$ we have $s/t < 1 + s/t \leq 2s/t$, so

$$\begin{aligned}\int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt &= \int_0^1 \left(1 + \frac{s}{t}\right)^{\beta_2} t^{\alpha-1}\,dt \\ &\simeq s^{\beta_2}\int_0^1 t^{\alpha-\beta_2-1}\,dt \\ &\simeq s^{\beta_2}\end{aligned}$$

for $s \geq 1$ if $\beta_2 < \alpha$ (where $F(s) \simeq G(s)$ for positive functions F and G on $[1, \infty)$ indicates the existence of positive constants C_1 and C_2 such that $C_1 F(s) \leq G(s) \leq C_2 F(s)$ for all $s \in [1, \infty)$), whereas the integrals diverge if $\beta_2 \geq \alpha$. Moreover, the expression

$$\int_0^1 \omega_2(s/t) \, \frac{\psi(t)}{t} \, dt = \int_0^1 \Big(1 + \frac{s}{t}\Big)^{\beta_2} t^{\alpha - 1} \, dt$$

defines a positive, continuous function of s on $\mathbb{R}^+$, so it follows that condition (C) is satisfied if and only if $\beta_2 \leq \beta_1$ and $\beta_2 < \alpha$.

(b): For $s \geq 1$ we have

$$\int_0^1 \omega_2(s/t) \, \frac{\psi(t)}{t} \, dt = \int_s^\infty \frac{\omega_2(x)}{x} \, \psi(s/x) \, dx = \int_s^\infty \frac{e^{-x}}{x} \, \frac{s^\alpha}{x^\alpha} \, dx \leq \int_s^\infty \frac{e^{-x}}{x} \, dx \leq \frac{e^{-s}}{s} .$$

Moreover,

$$\int_0^1 \omega_2(s/t) \, \frac{\psi(t)}{t} \, dt \leq \int_0^1 \frac{\psi(t)}{t} \, dt < \infty$$

for all $s \in \mathbb{R}^+$, so condition (C) is satisfied and U_ψ thus defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. On the other hand, since

$$\int_0^1 \omega_2(s/t) \, \frac{\psi(t)}{t} \, dt \geq \int_s^{2s} \frac{e^{-x}}{x} \, \frac{s^\alpha}{x^\alpha} \, dx \geq \frac{1}{2^{\alpha+1} s} \int_s^{2s} e^{-x} \, dx \geq \frac{1}{2^{\alpha+2}} \, \frac{e^{-s}}{s}$$

for $s \geq 1$, it is not possible to replace $\omega_1(x)$ by a function tending faster to zero at infinity.

(c): For $s \in \mathbb{R}^+$ we have

$$\int_0^1 \omega_2(s/t) \, \frac{\psi(t)}{t} \, dt = \int_s^\infty \frac{\omega_2(x)}{x} \, \psi(s/x) \, dx = \int_s^\infty \frac{e^{x - x^2/s^2}}{x} \, dx = \int_1^\infty \frac{e^{sy - y^2}}{y} \, dy.$$

Moreover, for $s \geq 4$

$$\int_{s/4}^\infty \frac{e^{sy - y^2}}{y} \, dy \leq \frac{4}{s} \int_{s/4}^\infty e^{-(y - s/2)^2 + s^2/4} \, dy = 4 \int_{-s/4}^\infty e^{-u^2} \, du \, \frac{e^{s^2/4}}{s}$$

and

$$\int_1^{s/4} \frac{e^{sy - y^2}}{y} \, dy \leq \int_1^{s/4} e^{sy} \, dy \leq \frac{e^{s^2/4}}{s},$$

so condition (C) is satisfied and U_ψ thus defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. On the other hand, the estimate

$$\int_0^1 \omega_2(s/t) \, \frac{\psi(t)}{t} \, dt = \int_1^\infty \frac{e^{sy - y^2}}{y} \, dy \geq \frac{1}{s} \int_{s/2}^{s/2+1} e^{-(y - s/2)^2 + s^2/4} \, dy = \int_0^1 e^{-u^2} \, du \, \frac{e^{s^2/4}}{s}$$

for $s \geq 2$ shows that it is not possible to replace $\omega_1(x)$ by a function tending slower to infinity at infinity. □

In Example 2.4(b) we have $\omega_2(x)/\omega_1(x) \to \infty$ as $x \to \infty$, which should be compared to the conclusion in Corollary 2.3(b). Conversely, Example 2.4(c) shows an example where we need $\omega_2(x)/\omega_1(x) \to 0$ rapidly as $x \to \infty$ in order for U_ψ to be defined.

3 Extensions to weighted spaces of measures

Identifying the dual space of $L^1(\omega)$ with $L^\infty(1/\omega)$ as in the introduction, we have the following result about the adjoint of U_ψ.

Proposition 3.1 *Let ψ be a non-negative, measurable function on $[0,1]$ and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator, and consider the adjoint operator $U_\psi^* : L^\infty(1/\omega_2) \to L^\infty(1/\omega_1)$.*

(a) For $h \in L^\infty(1/\omega_2)$ we have

$$(U_\psi^* h)(x) = \int_0^1 h(x/t)\,\frac{\psi(t)}{t}\,dt$$

for almost all $x \in \mathbb{R}^+$.

(b) U_ψ^* maps $C_0(1/\omega_2)$ into $C_0(1/\omega_1)$.

Proof (a): Let $h \in L^\infty(1/\omega_2)$. Since $|h(x/t)| \le \|h\|_{L^\infty(1/\omega_2)}\omega_2(x/t)$ for almost all $x, t \in \mathbb{R}^+$, it follows from condition (C) that $\int_0^1 h(x/t)\psi(t)/t\,dt$ is defined and satisfies

$$\left|\int_0^1 h(x/t)\,\frac{\psi(t)}{t}\,dt\right| \le \|h\|_{L^\infty(1/\omega_2)} \int_0^1 \omega_2(x/t)\,\frac{\psi(t)}{t}\,dt \le C\|h\|_{L^\infty(1/\omega_2)}\omega_1(x)$$

for almost all $x \in \mathbb{R}^+$. Hence the function $x \mapsto \int_0^1 h(x/t)\psi(t)/t\,dt$ belongs to $L^\infty(1/\omega_1)$. Also, for $f \in L^1(\omega_1)$ we have

$$\begin{aligned}\langle f, U_\psi^* h\rangle = \langle U_\psi f, h\rangle &= \int_0^\infty (U_\psi f)(s)h(s)\,ds \\ &= \int_0^\infty \int_0^s \frac{1}{s}\,f(x)\psi(x/s)h(s)\,dx\,ds \\ &= \int_0^\infty \int_x^\infty \frac{h(s)}{s}\,\psi(x/s)\,ds\,f(x)\,dx\end{aligned}$$

from which it follows that

$$(U_\psi^* h)(x) = \int_x^\infty \frac{h(s)}{s}\,\psi(x/s)\,ds = \int_0^1 h(x/t)\,\frac{\psi(t)}{t}\,dt$$

for almost all $x \in \mathbb{R}^+$.

(b): It suffices to show that U_ψ^* maps $C_c(\mathbb{R}^+)$ (the continuous functions on $\mathbb{R}^+$ with compact support) into $C_0(1/\omega_1)$. Let $g \in C_c(\mathbb{R}^+)$, let $x_0 \in \mathbb{R}^+$ and let (x_n) be a sequence in $\mathbb{R}^+$ with $x_n \to x_0$ as $n \to \infty$. Then

$$(U_\psi^* g)(x_n) - (U_\psi^* g)(x_0) = \int_0^1 (g(x_n/t) - g(x_0/t))\,\frac{\psi(t)}{t}\,dt$$

for $n \in \mathbb{N}$. Since g is bounded on $\mathbb{R}^+$ and since $\int_0^1 \psi(t)/t\,dt < \infty$ by Corollary 2.2, it follows from Lebesgue's dominated convergence theorem that $(U_\psi^* g)(x_n) \to (U_\psi^* g)(x_0)$ as $n \to \infty$. Hence $U_\psi^* g$ is continuous on $\mathbb{R}^+$. Finally, from the expression

$$(U_\psi^* g)(x) = \int_x^\infty \frac{g(s)}{s}\,\psi(x/s)\,ds$$

it follows that $\operatorname{supp} U_\psi^* g \subseteq \operatorname{supp} g$, so we conclude that $U_\psi^* g \in C_c(\mathbb{R}^+) \subseteq C_0(1/\omega_1)$. $\square$

Let V_ψ be the restriction of U_ψ^* to $C_0(1/\omega_2)$ considered as a map into $C_0(1/\omega_1)$. We then immediately have the following result.

Corollary 3.2 *Let ψ be a non-negative, measurable function on $[0,1]$ and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. The bounded operator $\overline{U}_\psi = V_\psi^*$ from $M(\omega_1)$ to $M(\omega_2)$ is an extension of U_ψ.*

Let ψ be a non-negative, continuous function on $[0,1]$ with $\psi(0) = 0$. For $\mu \in M(\omega_1)$ and $x > 0$ let

$$(W_\psi \mu)(x) = \frac{1}{x} \int_{(0,x)} \psi(s/x)\, d\mu(s).$$

Proposition 3.3 *Let ψ be a non-negative, continuous function on $[0,1]$ and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. Then $W_\psi \mu \in L^1(\omega_2)$ and*

$$\overline{U}_\psi \mu = W_\psi \mu + \int_0^1 \frac{\psi(t)}{t}\, dt \cdot \mu(\{0\}) \delta_0$$

for $\mu \in M(\omega_1)$. In particular $\operatorname{ran} \overline{U}_\psi \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0$ and $\overline{U}_\psi$ maps $M((0,\infty), \omega_1)$ into $L^1(\omega_2)$.

Proof By Corollary 2.2 we have $\int_0^1 \psi(t)/t\, dt < \infty$, so it follows that $\psi(0) = 0$. Let $\mu \in M(\omega_1)$ with $\mu(\{0\}) = 0$. By condition (C) we have

$$\begin{aligned}
\int_{(0,\infty)} \int_s^\infty \frac{1}{x} \psi(s/x) \omega_2(x)\, dx\, d|\mu|(s) &= \int_{(0,\infty)} \int_0^1 \omega_2(s/t) \frac{\psi(t)}{t}\, dt\, d|\mu|(s) \\
&\le C \int_{(0,\infty)} \omega_1(s)\, d|\mu|(s) = C \|\mu\|_{M(\omega_1)} < \infty,
\end{aligned}$$

so it follows from Fubini's theorem that

$$\int_0^\infty \frac{1}{x} \int_{(0,x)} \psi(s/x)\, d|\mu|(s)\, \omega_2(x)\, dx < \infty.$$

Hence $W_\psi \mu \in L^1(\omega_2)$. Moreover, for $g \in C_0(1/\omega_2)$ we have

$$\begin{aligned}
\langle g, \overline{U}_\psi \mu \rangle = \langle V_\psi g, \mu \rangle &= \int_{(0,\infty)} \int_0^1 g(s/t) \frac{\psi(t)}{t}\, dt\, d\mu(s) \\
&= \int_{(0,\infty)} \int_s^\infty \frac{g(x)}{x} \psi(s/x)\, dx\, d\mu(s) \\
&= \int_0^\infty \frac{1}{x} \int_{(0,x)} \psi(s/x)\, d\mu(s)\, g(x)\, dx \\
&= \int_0^\infty (W_\psi \mu)(x) g(x)\, dx = \langle g, W_\psi \mu \rangle,
\end{aligned}$$

so we conclude that $\overline{U}_\psi \mu = W_\psi \mu$. Finally, for $g \in C_0(1/\omega_2)$ we have

$$\langle g, \overline{U}_\psi \delta_0 \rangle = \langle V_\psi g, \delta_0 \rangle = (V_\psi g)(0) = g(0) \int_0^1 \frac{\psi(t)}{t}\, dt = \langle g, \int_0^1 \frac{\psi(t)}{t}\, dt \cdot \delta_0 \rangle.$$

Since $W_\psi \delta_0 = 0$ this finishes the proof. $\square$

The conclusion about the range of $\overline{U}_\psi$ can be generalized to the case, where ψ is not assumed to be continuous.

Proposition 3.4 *Let ψ be a non-negative, measurable function on $[0,1]$ and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. Then* $\operatorname{ran} \overline{U}_\psi \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0$.

Proof Choose a sequence of non-negative, continuous functions (ψ_n) on $[0,1]$ with $\psi_n \leq \psi$ and

$$\int_0^1 \frac{\psi(t) - \psi_n(t)}{t}\, dt \to 0 \qquad \text{as } n \to \infty.$$

For $\mu \in M(\omega_1)$ and $g \in C_0(1/\omega_2)$ we have

$$\begin{aligned}
|\langle g, (\overline{U}_\psi - \overline{U}_{\psi_n})\mu \rangle| &= |\langle (V_\psi - V_{\psi_n}) g, \mu \rangle| \\
&= \left| \int_{\mathbb{R}^+} \int_0^1 g(x/t)\, \frac{\psi(t) - \psi_n(t)}{t}\, dt\, d\mu(x) \right| \\
&\leq \|g\|_{C_0(1/\omega_2)} \int_{\mathbb{R}^+} \int_0^1 \omega_2(x/t)\, \frac{\psi(t) - \psi_n(t)}{t}\, dt\, d|\mu|(x).
\end{aligned}$$

Let

$$p_n(x) = \int_0^1 \omega_2(x/t)\, \frac{\psi(t) - \psi_n(t)}{t}\, dt$$

for $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$. By condition (C) there exists a constant C such that $p_n(x) \leq C\omega_1(x)$ for every $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Moreover, for every $x \in \mathbb{R}^+$ we have $p_n(x) \to 0$ as $n \to \infty$ by Lebesgue's dominated convergence theorem. Hence

$$\|(\overline{U}_\psi - \overline{U}_{\psi_n})\mu\|_{M(\omega_2)} = \sup_{\|g\|_{C_0(1/\omega_2)} \leq 1} |\langle g, (\overline{U}_\psi - \overline{U}_{\psi_n})\mu \rangle| \leq \int_{\mathbb{R}^+} p_n(x)\, d|\mu|(x) \to 0$$

as $n \to \infty$ again by Lebesgue's dominated convergence theorem. Consequently, $\overline{U}_{\psi_n} \to \overline{U}_\psi$ strongly as $n \to \infty$. Since $\operatorname{ran} \overline{U}_{\psi_n} \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0$ for $n \in \mathbb{N}$ by Proposition 3.3, the same thus holds for $\operatorname{ran} \overline{U}_\psi$. $\square$

Corollary 3.5 *Let ψ be a non-negative, measurable function on $[0,1]$ and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. For $s > 0$ we then have $(\overline{U}_\psi \delta_s)(x) = \psi(s/x)/x$ for almost all $x \geq s$ and $(\overline{U}_\psi \delta_s)(x) = 0$ for almost all $x < s$.*

Proof For ψ continuous, this follows from Proposition 3.3. For general ψ it follows from the approach in the proof of Proposition 3.4 using $\overline{U}_{\psi_n} \to \overline{U}_\psi$ strongly as $n \to \infty$. $\square$

It follows from Corollary 3.5 that

$$\|\overline{U}_\psi \delta_s\|_{M(\omega_2)} = \int_s^\infty \frac{\omega_2(x)}{x}\,\psi(s/x)\,dx = \int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt,$$

whereas $\|\delta_s\|_{M(\omega_1)} = \omega_1(s)$. Since $\overline{U}_\psi$ is bounded we thus recover condition (C). If we without using Theorem 2.1 could show that if $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator, then is has a bounded extension $\overline{U}_\psi : M(\omega_1) \to M(\omega_2)$ for which Corollary 3.5 holds, then we would in this way obtain an alternative proof of condition (C).

4 Weakly compact operators

We finish the paper by showing that there are no non-zero, weakly compact generalized Hardy-Cesàro operators between $L^1(\omega_1)$ and $L^1(\omega_2)$.

Proposition 4.1 *Let ψ be a non-negative, measurable function on $[0,1]$ and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. If $\psi \neq 0$, then U_ψ is not weakly compact.*

Proof For $f \in L^1(\omega_1)$ and $x \in \mathbb{R}^+$ we have

$$(U_\psi f)(x) = \frac{1}{x}\int_0^x f(s)\psi(s/x)\,ds = \int_0^\infty f(s)\rho(s)(x)\omega_1(s)\,ds,$$

where (with a slight change of notation compared to the proof of Theorem 2.1)

$$\rho(s)(x) = \frac{1}{\omega_1(s)}\,\frac{1}{x}\,\psi(s/x)\,1_{x\geq s}$$

for $x, s \in \mathbb{R}^+$. In the proof of Theorem 2.1 we saw that $\rho(s) \in L^1(\omega_2)$ with $\|\rho(s)\|_{L^1(\omega_2)} \leq C$ for a constant C for almost all $s \in \mathbb{R}^+$. It thus follows from [4, Theorem VI.8.10] that U_ψ is weakly compact if and only if $\{\rho(s) : s \in \mathbb{R}^+\}$ is contained in a weakly compact set of $L^1(\omega_2)$ (except possibly for s belonging to a null-set). Consider $\rho(s)$ as an element of $C_0(1/\omega_2)^*$ for $s \in \mathbb{R}^+$ and let $g \in C_0(1/\omega_2)$. Then

$$\begin{aligned}\langle g, \rho(s)\rangle &= \int_0^\infty g(x)\rho(s)(x)\,dx \\ &= \frac{1}{\omega_1(s)}\int_s^\infty \frac{g(x)}{x}\,\psi(s/x)\,dx \\ &= \frac{1}{\omega_1(s)}\int_0^1 g(s/t)\,\frac{\psi(t)}{t}\,dt.\end{aligned}$$

Since $g(s/t) \to g(0)$ as $s \to 0_+$ for all $t > 0$, it follows from Lebesgue's dominated convergence theorem that

$$\langle g, \rho(s)\rangle \to \frac{1}{\omega_1(0)}\,g(0)\int_0^1 \frac{\psi(t)}{t}\,dt$$

as $s \to 0_+$. We therefore conclude that

$$\rho(s) \to \frac{1}{\omega_1(0)} \int_0^1 \frac{\psi(t)}{t}\, dt \cdot \delta_0$$

weak-star in $M(\omega_2)$ as $s \to 0_+$. Since $\delta_0 \notin L^1(\omega_2)$, it follows that $\{\rho(s) : s \in \mathbb{R}^+\}$ is not contained in a weakly compact set of $L^1(\omega_2)$ (even excepting null sets), and the result follows. $\square$

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